Fermionic R-Operator and Integrability of the One-Dimensional Hubbard Model

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We propose a new type of the Yang-Baxter equation (YBE) and the decorated Yang-Baxter equation (DYBE). Those relations for the fermionic R-operator were introduced recently as a tool to treat the integrability of the fermion models. Using the YBE and the DYBE for the XX fermion model, we construct the fermionic R-operator for the one-dimensional (1D) Hubbard model. It gives another proof of the integrability of the 1D Hubbard model. Furthermore a new approach to the $SO(4)$ symmetry of the 1D Hubbard model is discussed.

1 Introduction

There have been much interests in strongly correlated electron systems. Among the exactly solvable ones, the one-dimensional (1D) Hubbard model,

$$\mathcal{H} = -\sum_{j=1}^{N}\sum_{\sigma=\uparrow\downarrow}(c_{j+1\sigma}^{\dagger}c_{j\sigma} + c_{j\sigma}^{\dagger}c_{j+1\sigma}) + U\sum_{j=1}^{N}(n_{j\uparrow} - \frac{1}{2})(n_{j\downarrow} - \frac{1}{2}). \tag{1.1}$$

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is the most interesting model. Here $c^\dagger_{j\sigma}$ and $c_{j\sigma}$ are the fermion creation and annihilation operators and $n_{j\sigma}$ is the fermion number operator,

$$n_{j\sigma} = c^\dagger_{j\sigma} c_{j\sigma}. \tag{1.2}$$

The parameter U is the coupling constant describing the Coulomb interaction between electrons with opposite spins on the same site. Lieb and Wu [1] diagonalized the Hamiltonian (1.1) by means of the coordinate Bethe ansatz method under the periodic boundary condition (PBC),

$$c^\dagger_{N+1,\sigma} = c^\dagger_{1\sigma}, \quad c_{N+1,\sigma} = c_{1\sigma}, \qquad \sigma = \uparrow\downarrow . \tag{1.3}$$

The bulk properties of the 1D Hubbard model have been investigated by analyzing the associated Bethe ansatz equations. [2]

The integrability of the 1D Hubbard model has been studied by Shastry [3–5] and Wadati *et al.*. [6–8] Shastry introduced the Jordan-Wigner transformation, [9]

$$c_{j\uparrow} = \exp\left\{ i\pi \sum_{k=1}^{j-1}(n_{k\uparrow} - 1) \right\} \sigma^-_j ,$$

$$c_{j\downarrow} = \exp\left\{ i\pi \sum_{k=1}^{j-1}(n_{k\downarrow} - 1) \right\} \exp\left\{ i\pi \sum_{k=1}^{N}(n_{k\uparrow} - 1) \right\} \tau^-_j , \tag{1.4}$$

to change the fermionic Hamiltonian (1.1) into an equivalent coupled spin model,

$$H = \sum_{j=1}^{N}(\sigma^+_{j+1}\sigma^-_j + \sigma^+_j\sigma^-_{j+1}) + \sum_{j=1}^{N}(\tau^+_{j+1}\tau^-_j + \tau^+_j\tau^-_{j+1}) + \frac{U}{4}\sum_{j=1}^{N}\sigma^z_j\tau^z_j, \tag{1.5}$$

where σ and τ are two species of the Pauli matrices commuting each other, and

$$\sigma^\pm_j = \frac{1}{2}(\sigma^x_j \pm i\sigma^y_j), \quad \tau^\pm_j = \frac{1}{2}(\tau^x_j \pm i\tau^y_j). \tag{1.6}$$

Shastry constructed the L-operator and R-matrix which satisfy the Yang-Baxter relation,

$$R_{12}(u_1, u_2)L_{a1}(u_1)L_{a2}(u_2) = L_{a2}(u_2)L_{a1}(u_1)R_{12}(u_1, u_2). \tag{1.7}$$

In ref. [5], the decorated Yang-Baxter equation (DYBE) for the free-fermion model was used to prove the Yang-Baxter relation for the coupled spin model, (1.7). It is a relation similar to, but not equivalent to the Yang-Baxter equation (YBE), Recently Shiroishi and Wadati [10,11] found that the YBE and the DYBE for the free-fermion model can be generalized to a larger set of relations among the L-operators called tetrahedral Zamolodchikov algebra (TZA). [12] By use of the TZA, the Yang-Baxter equation for the R-matrix,

$$R_{12}(u_1, u_2)R_{13}(u_1, u_3)R_{23}(u_2, u_3) = R_{23}(u_2, u_3)R_{13}(u_1, u_3)R_{12}(u_1, u_2) \tag{1.8}$$

can be proved. [10, 11, 13]

The coupled spin model (1.5) is often referred to as the 1D Hubbard model, since it is related to the fermionic Hamiltonian (1.1) through the Jordan-Wigner transformation (1.4). However, because of the non-locality of the Jordan-Wigner transformation (1.4), the PBC for the fermion model (1.3) does not correspond to the PBC for the coupled spin model. Moreover the $SO(4)$ symmetry [14–18], which is one of the most important properties of the Hamiltonian (1.1), is not clear in the coupled spin model (1.5).

To discuss the fermionic Hamiltonian directly, we can employ the fermionic formulation of the Yang-Baxter relation (graded Yang-Baxter relation). [7] In this formulation, the transfer matrix that respects the PBC (1.3) can be constructed by taking the supertrace of the monodromy matrix. Recently it has been shown that the transfer matrix naturally enjoys the $SO(4)$ symmetry. [19, 20] Based on this fermionic formulation, the algebraic Bethe ansatz method has been developed by Ramos and Martins. [21, 22]

The aim of this paper is to present yet another approach to discuss the integrability of the 1D Hubbard model using the fermionic R-operator. The notion of the fermionic R-operator was introduced in relation to the XXZ fermion model. [23, 24] The fermionic R-operator consists of the fermion operators and is a solution of the YBE.

In this paper, we first consider the XYZ fermion model. The fermionic R-operator for the XYZ fermion model, which satisfies the YBE, is explicitly obtained. The YBE ensures the commutativity of the transfer operators. The logarithmic derivative of the transfer operator yields the Hamiltonian of the XYZ fermion model under the PBC.

Next we consider the DYBE for the fermionic R-operator and find that the fermionic R-operator related to the XY fermion model fulfills the DYBE. Combining the YBE and the DYBE of the fermionic R-operator for the XX fermion models, we construct a new fermionic R-operator and an L-operator for the 1D Hubbard model. The transfer operator constructed from the L-operator generates the Hamiltonian and other conserved operators under the PBC.

We also prove the YBE for the new integrable R-operator by considering the fermionic variant of the tetrahedral Zamolodchikov algebra (TZA). Then we consider the generalized transfer operator which is constructed from the fermionic R-operator. A new fermionic Hamiltonian is obtained from the transfer operator.

It is possible to discuss the $SO(4)$ symmetry of the 1D Hubbard model using the fermionic R-operator. The (anti) commutation relations between the fermionic R-operator and the generators of the $SO(4)$ algebra are obtained. Those relations are extended to the monodromy operator and the transfer operator. We can show the $SO(4)$ invariance of the local conserved operators, though the transfer operator itself is not $SO(4)$ invariant.

The outline of this paper is as follows. In §2, we study the Yang-Baxter equation (YBE) for the XYZ fermion model. Fermionic R-operator related to the XYZ fermion model is explicitly shown by use of Baxter's parameterization. In §3, the decorated Yang-Baxter equation (DYBE) for the fermionic R-operator is introduced. It is necessary to impose the free-fermion condition on the Boltzmann weights in order that the DYBE is fulfilled. Some connections to the XY fermion model with an external field are also discussed. In §4, we discuss the integrability of the 1D Hubbard model. The fermionic R-operator for the 1D

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Hubbard model is constructed from the R-operators related to the XX fermion models with spin up and down. The YBE and the DYBE are utilized to show the Yang-Baxter relation, where a new L-operator is also introduced. In §5, we prove the YBE for the fermionic R-operator of the Hubbard model. We use the fermionic variant of the TZA. The discussions proceed just in parallel with the case of the coupled spin model (1.5). We also use a generalized transfer operator which consists of the fermionic R-operators. In §6, we give a new approach to the $SO(4)$ symmetry of the 1D Hubbard model. The fermionic R-operator has a natural $SO(4)$ symmetry. We show that all the local fermionic conserved operators derived from the transfer operator are $SO(4)$ invariant. The last section is devoted to the concluding remarks.

2 Fermionic R-Operator for the XYZ Fermion Model

It is well understood that the Yang-Baxter equation (YBE)

$$\mathcal{R}_{12}(u-v)\mathcal{R}_{13}(u)\mathcal{R}_{23}(v) = \mathcal{R}_{23}(v)\mathcal{R}_{13}(u)\mathcal{R}_{12}(u-v), \tag{2.1}$$

plays the most important role in the quantum integrable models. [25–29]

We look for a solution of the YBE (2.1) in the form

$$\begin{aligned}
\mathcal{R}_{jk}(u) =& a(u)\{-n_j n_k + (1-n_j)(1-n_k)\} + b(u)\{n_j(1-n_k) + (1-n_j)n_k\} \\
&+ c(u)(c_j^\dagger c_k + c_k^\dagger c_j) - d(u)(c_j^\dagger c_k^\dagger + c_j c_k),
\end{aligned} \tag{2.2}$$

where $c_j^\dagger$ and c_j are the fermion creation annihilation operators satisfying the canonical anti-commutation relations

$$\{c_j, c_k\} = \{c_j^\dagger, c_k^\dagger\} = 0, \qquad \{c_j^\dagger, c_k\} = \delta_{jk}, \tag{2.3}$$

and $n_j = c_j^\dagger c_j$.

Substituting (2.2) into (2.1) and using (2.3), we get the following relations among the functions $a(u)$, $b(u)$, $c(u)$ and $d(u)$,

$$\begin{aligned}
a(u-v)c(u)a(v) + d(u-v)a(u)d(v) &= b(u-v)c(u)b(v) + c(u-v)a(u)c(v), \\
a(u-v)b(u)c(v) + d(u-v)d(u)b(v) &= b(u-v)a(u)c(v) + c(u-v)c(u)b(v), \\
c(u-v)b(u)a(v) + b(u-v)d(u)d(v) &= c(u-v)a(u)b(v) + b(u-v)c(u)c(v), \\
a(u-v)d(u)b(v) + d(u-v)b(u)c(v) &= b(u-v)d(u)a(v) + c(u-v)b(u)d(v), \\
a(u-v)a(u)d(v) + d(u-v)c(u)a(v) &= b(u-v)b(u)d(v) + c(u-v)d(u)a(v), \\
d(u-v)a(u)a(v) + a(u-v)c(u)d(v) &= d(u-v)b(u)b(v) + a(u-v)d(u)c(v).
\end{aligned} \tag{2.4}$$

These are nothing but the relations among the Boltzmann weights of the 8-vertex model (Baxter model). [25] Then, using Baxter's parameterization, we can solve (2.4) as follows,

$$\begin{aligned}
a(u) &= \mathrm{sn}(u+2\eta), \quad b(u) = \mathrm{sn}(u), \quad c(u) = \mathrm{sn}(2\eta), \\
d(u) &= ka(u)b(u)c(u) = k\,\mathrm{sn}(2\eta)\,\mathrm{sn}(u)\,\mathrm{sn}(u+2\eta),
\end{aligned} \tag{2.5}$$

where $\operatorname{sn} u = \operatorname{sn}(u, k)$ is Jacobi's elliptic function with the modulus k and η is the anisotropy parameter.

Thus we have found a new type of solution for the YBE (2.1). We call (2.2) the fermionic R-operator (related to the XYZ fermion model). We remark that Destri and Segalini [23] have discussed the special case $(k = 0)$ of (2.2). The fermionic R-operator (2.2) can also be obtained from the R-matrix of the XYZ spin model (Appendix A).

It is remarkable that $\mathcal{R}_{jk}(u)$ has a property,

$$\mathcal{R}_{jk}(0) = \operatorname{sn} 2\eta \mathcal{P}_{jk}, \tag{2.6}$$

where $\mathcal{P}_{jk}$ is the permutation operator,

$$
\begin{aligned}
\mathcal{P}_{jk} &= 1 - (c_j^\dagger - c_k^\dagger)(c_j - c_k) = 1 - n_j - n_k + c_j^\dagger c_k + c_k^\dagger c_j, \\
\mathcal{P}_{jk} c_j &= c_k \mathcal{P}_{jk}, \qquad \mathcal{P}_{jk} c_j^\dagger = c_k^\dagger \mathcal{P}_{jk}.
\end{aligned} \tag{2.7}
$$

This property is useful when we derive a related Hamiltonian from the transfer operator.

Unlike the R-matrix of the usual XYZ spin model, the fermionic R-operator is not symmetric, $\mathcal{R}_{12}(u) \neq \mathcal{R}_{21}(u)$. Instead, the following relation holds

$$\mathcal{R}_{12}(u; k) = \mathcal{R}_{21}(u; -k), \tag{2.8}$$

where we write the modulus k dependence explicitly. The fermionic R-operator (2.2) also has the unitarity relation in the form

$$\mathcal{R}_{12}(u)\mathcal{R}_{21}(-u) = \left(-\operatorname{sn}^2 u + \operatorname{sn}^2 2\eta\right) \mathbf{1}, \tag{2.9}$$

as well as the property

$$[\mathcal{R}_{jk}(u), (1 - 2n_j)(1 - 2n_k)] = 0. \tag{2.10}$$

Now we introduce the monodromy operator

$$\mathcal{T}_a(u) = \mathcal{R}_{aN}(u) \dots \mathcal{R}_{a1}(u), \tag{2.11}$$

where the suffix a means the auxiliary space, The transfer operator is given by the "supertrace" of the monodromy operator

$$\tau(u) = \operatorname{Str}_a \mathcal{T}_a(u). \tag{2.12}$$

Here we define the supertrace Str as

$$\operatorname{Str}_a X = {}_a\langle 0|X|0\rangle_a - {}_a\langle 1|X|1\rangle_a, \tag{2.13}$$

with the auxiliary fermion Fock space being

$$c_a|0\rangle_a = 0, \quad |1\rangle_a = c_a^\dagger|0\rangle_a, \quad {}_a\langle 0| = (|0\rangle_a)^\dagger, \quad {}_a\langle 1| = {}_a\langle 0|c_a, \tag{2.14}$$

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$$_a\langle 0|0\rangle_a = {}_a\langle 1|1\rangle_a = 1. \tag{2.15}$$

When we write the monodromy operator $\mathcal{T}_a(u)$ as

$$\mathcal{T}_a(u) = A(u)(1 - n_a) + B(u)c_a + C(u)c_a^\dagger + D(u)n_a, \tag{2.16}$$

the transfer operator (2.12) is given by

$$\tau(u) = A(u) - D(u). \tag{2.17}$$

The YBE for the fermionic R-operator leads to the global Yang-Baxter relation for the monodromy operator (2.11),

$$\mathcal{R}_{12}(u - v)\mathcal{T}_1(u)\mathcal{T}_2(v) = \mathcal{T}_2(v)\mathcal{T}_1(u)\mathcal{R}_{12}(u - v). \tag{2.18}$$

Rewriting (2.18) as

$$\mathcal{R}_{12}(u - v)\mathcal{T}_1(u)\mathcal{T}_2(v)\mathcal{R}_{12}^{-1}(u - v) = \mathcal{T}_2(v)\mathcal{T}_1(u), \tag{2.19}$$

and taking the supertrace (2.13) of both sides, we can show the commutativity of the transfer operators

$$[\tau(u), \tau(v)] = 0, \tag{2.20}$$

which ensures the existence of enough number of the fermionic conserved operators. In this way we can establish the integrability of the fermion model by means of the fermionic R-operator.

Using (2.6) in (2.12) with (2.11), we have

$$\begin{aligned}
\tau(0) &= (\operatorname{sn} 2\eta)^N \ \operatorname{Str}_a(\mathcal{P}_{aN} \ldots \mathcal{P}_{a2}\mathcal{P}_{a1}) \\
&= (\operatorname{sn} 2\eta)^N \ \mathcal{P}_{12}\mathcal{P}_{23} \ldots \mathcal{P}_{N-1,N},
\end{aligned} \tag{2.21}$$

where we have used some fundamental properties of the permutation operators

$$\begin{aligned}
\mathcal{P}_{aj}\mathcal{P}_{ak} &= \mathcal{P}_{jk}\mathcal{P}_{aj}, \qquad \mathcal{P}_{jk} = \mathcal{P}_{kj}, \\
\operatorname{Str}_a \mathcal{P}_{aj} &= 1.
\end{aligned} \tag{2.22}$$

Thus we find that $\tau(0)$ is proportional to the (left) shift operator $\hat{U}$ for fermions,

$$\hat{U} = \mathcal{P}_{12}\mathcal{P}_{23} \ldots \mathcal{P}_{N-1,N}. \tag{2.23}$$

It is easy to confirm that the shift operator $\hat{U}$ acts on the fermion operators as

$$\begin{aligned}
\hat{U}c_j &= c_{j+1}\hat{U}, \qquad\qquad j = 1, \ldots, N - 1, \\
\hat{U}c_N &= c_1\hat{U}.
\end{aligned} \tag{2.24}$$

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In a similar way as the spin models, the Hamiltonian of the XYZ fermion model is derived from the logarithmic derivative of the transfer operator

$$\mathcal{H}^{\text{XYZ}} = \tau(0)^{-1} \frac{\mathrm{d}}{\mathrm{d}u} \tau(u)\Big|_{u=0}$$
$$= \sum_{j=1}^{N} \mathcal{H}^{\text{XYZ}}_{j,j+1}, \tag{2.25}$$

where the two-point Hamiltonian density is given by

$$\mathcal{H}^{\text{XYZ}}_{j,j+1} = \frac{1}{\operatorname{sn} 2\eta} \Big\{ a'(0) \left(n_j n_{j+1} + (1-n_j)(1-n_{j+1}) \right)$$
$$+ b'(0)(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) + d'(0)(c_j^\dagger c_{j+1}^\dagger - c_j c_{j+1}) \Big\}. \tag{2.26}$$

Here and hereafter, $'$ means the derivative with respect to the spectral parameter u,

$$a'(0) = \operatorname{cn} 2\eta \ \operatorname{dn} 2\eta, \quad b'(0) = 1, \quad d'(0) = k \ \operatorname{sn}^2 2\eta. \tag{2.27}$$

In (2.25), we have used a fact that $\tau(0)^{-1}$ is proportional to the shift operator. In terms of an equivalent R-operator $\check{\mathcal{R}}_{jk}(u) \equiv \mathcal{P}_{jk}\mathcal{R}_{jk}(u)$ (c.f. Appendix A), the two-point Hamiltonian density is expressed as

$$\mathcal{H}^{\text{XYZ}}_{jk} = \frac{\mathrm{d}}{\mathrm{d}u} \check{R}_{jk}(u)\Big|_{u=0}. \tag{2.28}$$

One may wonder if our discussion using the fermionic R-operator goes fully in parallel with the standard treatment of the integrable spin models and no new aspect appears. In fact we can proceed the discussion almost in the same way as the spin models. However we like to point out an important difference between the spin model and the fermion model here. While the Hamiltonian (2.26) is hermitian, it is not invariant under the space inversion

$$j \longrightarrow N - j + 1. \tag{2.29}$$

It is easy to see that the modulus k in the Hamiltonian (2.26) changes its sign under the transformation (2.29). This point makes a difference between the XYZ spin model and the XYZ fermion model. Actually it is a consequence of the property (2.8).

In the case of $k = 0$, the fermionic R-operator (2.2) reduces to the one with

$$a(u) = \sin(u + 2\eta) \quad b(u) = \sin u \quad c(u) = \sin 2\eta \quad d(u) = 0, \tag{2.30}$$

which we have used in ref. [24] to discuss the integrability of the XXZ fermion model

$$\mathcal{H}^{\text{XXZ}} = \frac{1}{\sin 2\eta} \sum_{j=1}^{N} \Big\{ c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j + \cos 2\eta \left(n_j n_{j+1} + (1-n_j)(1-n_{j+1}) \right) \Big\}. \tag{2.31}$$

In this case, the fermionic R-operator becomes symmetric and the Hamiltonian (2.31) is invariant under the space inversion (2.29).

We note that the periodic boundary condition (PBC) for the fermion operators is satisfied in (2.26),

$$c_{N+1} = c_1, \qquad c_{N+1}^\dagger = c_1^\dagger. \tag{2.32}$$

It is sometimes necessary to consider the anti-periodic boundary condition (APBC) for the fermion operators

$$c_{N+1} = -c_1, \qquad c_{N+1}^\dagger = -c_1^\dagger. \tag{2.33}$$

To explain this, we consider the following twisted monodromy operator

$$\begin{aligned}
\tilde{\mathcal{T}}_a(u) &= (1 - 2n_a)\mathcal{T}_a(u) \\
&= (1 - 2n_a)\mathcal{R}_{aN}(u) \dots \mathcal{R}_{a1}(u).
\end{aligned} \tag{2.34}$$

Then, thanks to the property (2.10), the global Yang-Baxter relation for the twisted monodromy operator holds,

$$\mathcal{R}_{12}(u - v)\tilde{\mathcal{T}}_1(u)\tilde{\mathcal{T}}_2(v) = \tilde{\mathcal{T}}_2(v)\tilde{\mathcal{T}}_1(u)\mathcal{R}_{12}(u - v), \tag{2.35}$$

which leads to the commutativity of the twisted transfer operator $\tilde{\tau}(u) = \mathrm{Str}_a \, \tilde{\mathcal{T}}_a(u)$,

$$[\tilde{\tau}(u), \tilde{\tau}(v)] = 0. \tag{2.36}$$

We can show that the twisted transfer operator generates the Hamiltonian (2.26) under the APBC (2.33). In fact the logarithmic derivative of $\tilde{\tau}(u)$ yields a Hamiltonian

$$\begin{aligned}
\tilde{\mathcal{H}}^{\mathrm{XYZ}} =& \tilde{\tau}(0)^{-1} \frac{\mathrm{d}}{\mathrm{d}u} \tilde{\tau}(u) \Big|_{u=0} \\
=& \sum_{j=1}^{N-1} \mathcal{H}_{j,j+1}^{\mathrm{XYZ}} + \frac{1}{\mathrm{sn}\, 2\eta} \Big\{ a'(0) \left(n_N n_1 + (1 - n_N)(1 - n_1)\right) \\
& \qquad\qquad - b'(0)(c_N^\dagger c_1 + c_1^\dagger c_N) - d'(0)(c_N^\dagger c_1^\dagger - c_N c_1) \Big\}.
\end{aligned} \tag{2.37}$$

Here we have used a relation

$$(1 - 2n_N) \, \mathrm{Str}_a \left\{ (1 - 2n_a)\mathcal{P}_{aN}\mathcal{H}_{a1}^{XYZ} \right\} = (1 - 2n_N)\mathcal{H}_{N1}^{XYZ}(1 - 2n_N), \tag{2.38}$$

as well as the identities

$$(1 - 2n_N)c_N(1 - 2n_N) = -c_N, \quad (1 - 2n_N)c_N^\dagger(1 - 2n_N) = -c_N^\dagger. \tag{2.39}$$

Note that if we express $\mathcal{T}_a(u)$ as (2.16), the twisted transfer operator takes the following form, [23]

$$\tilde{\tau}(u) = A(u) + D(u). \tag{2.40}$$

3 Decorated Yang-Baxter Equation

The decorated Yang-Baxter equation (DYBE) is an algebraic relation similar to the YBE (2.1), but not equivalent to the YBE. Shastry [5] showed that the free-fermion condition on the Boltzmann weights is necessary so that the solution of the YBE also satisfies the DYBE. [5,11] The DYBE is considered to be a hidden algebraic structure of the free-fermion model (equivalently, XY model).

We discuss the DYBE in the context of the fermionic R-operator. The DYBE for the fermionic R-operator is defined by

$$\mathcal{R}_{12}(u+v)(2n_1-1)\mathcal{R}_{13}(u)\mathcal{R}_{23}(v) = \mathcal{R}_{23}(v)\mathcal{R}_{13}(u)(2n_1-1)\mathcal{R}_{12}(u+v). \tag{3.1}$$

The DYBE (3.1) looks alike the YBE (2.1), but the operator $(2n_1-1)$ inserted in both sides can not be absorbed by the redefinition of $R_{jk}(u)$. It is possible to simplify the DYBE by use of the YBE as follows.

First, by multiplying $(2n_1-1)(2n_2-1)$ to the YBE (2.1) from the left and $(2n_3-1)$ from the right, we get a relation

$$\begin{aligned}
&\mathcal{R}_{12}(u-v)(2n_1-1)\mathcal{R}_{13}(u)(2n_2-1)\mathcal{R}_{23}(v)(2n_3-1) \\
&= (2n_2-1)\mathcal{R}_{23}(v)(2n_3-1)\mathcal{R}_{13}(u)(2n_1-1)\mathcal{R}_{12}(u-v),
\end{aligned} \tag{3.2}$$

where we have used a general property (2.10) and an equivalent relation,

$$(2n_j-1)\mathcal{R}_{jk}(u)(2n_k-1) = (2n_k-1)\mathcal{R}_{jk}(u)(2n_j-1). \tag{3.3}$$

Then comparing eq. (3.2) with the definition of the DYBE (3.1), we find a condition

$$(2n_j-1)\mathcal{R}_{jk}(u)(2n_k-1) = \mathcal{R}_{jk}(-u). \tag{3.4}$$

Hence we see that the DYBE is equivalent to the condition (3.4) when we have the YBE. In terms of the Boltzmann weights, the condition (3.4) is rephrased as

$$a(u) = a(-u), \quad -b(u) = b(-u), \quad c(u) = c(-u), \quad -d(u) = d(-u). \tag{3.5}$$

Using the explicit parameterization (2.5), we find that the condition (3.5) corresponds to

$$2\eta = K, \quad 3K, \tag{3.6}$$

where K is the complete elliptic integral of the first kind. In this case, the free-fermion condition

$$a^2(u) + b^2(u) = c^2(u) + d^2(u), \tag{3.7}$$

is fulfilled. Thus as in the case of the spin models, we have shown that the free-fermion condition is required for the validity of the DYBE (3.1).

Hereafter we restrict our consideration to the case $2\eta = K$. Then, from (2.5), the Boltzmann weights $a(u), b(u), c(u)$ and $d(u)$ are explicitly given as

$$a(u) = \operatorname{cd} u, \quad b(u) = \operatorname{sn} u, \quad c(u) = 1,$$
$$d(u) = k \operatorname{cd} u \ \operatorname{sn} u, \tag{3.8}$$

where $\operatorname{cd} u = \operatorname{cn} u/ \operatorname{dn} u$. The corresponding Hamiltonian is the XY fermion model

$$\mathcal{H}^{\mathrm{XY}} = \sum_{j=1}^{N} \left\{ c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j + k \left(c_j^\dagger c_{j+1}^\dagger - c_j c_{j+1} \right) \right\}. \tag{3.9}$$

In the special case $k = 0$, this Hamiltonian reduces to the XX fermion model

$$\mathcal{H}^{\mathrm{XX}} = \sum_{j=1}^{N} (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j). \tag{3.10}$$

As an interesting application of the YBE (2.1) and the DYBE (3.1), we shall prove the integrability of the XY fermion model in an external field,

$$\mathcal{H}^{\mathrm{XY}\mu} = \sum_{j=1}^{N} \left\{ c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j + k \left(c_j^\dagger c_{j+1}^\dagger - c_j c_{j+1} \right) + \mu \left(n_j - \frac{1}{2} \right) \right\}. \tag{3.11}$$

The external field μ can be regarded as the chemical potential for the XY fermion model. Since the term $\sum_{j=1}^{N}(n_j - 1/2)$ does not commute with the Hamiltonian (3.9) unless $k = 0$, the addition of the extra term is not trivial at all. It is well known that the model (3.11) can be diagonalized by means of the Fourier transformation through the Bogoliubov transformation. [30] The case $k = \pm 1$ corresponds to the transverse Ising model. [31]

The following argument is in common with the one explored for the spin models. [3,11] From now on, let $\mathcal{R}_{jk}(u)$ denote the fermionic R-operator for the XY fermion model. Taking a linear combination of the YBE (2.1) and the DYBE (3.1), we have

$$\left\{ \alpha \mathcal{R}_{12}(u - v) + \beta \mathcal{R}_{12}(u + v)(2n_1 - 1) \right\} \mathcal{R}_{13}(u)\mathcal{R}_{23}(v)$$
$$= \mathcal{R}_{23}(v)\mathcal{R}_{13}(u) \left\{ \alpha \mathcal{R}_{12}(u - v) + \beta(2n_1 - 1)\mathcal{R}_{12}(u + v) \right\}, \tag{3.12}$$

where α and β are not operators but constants.

Now we look for a solution of the Yang-Baxter relation

$$\mathcal{R}_{12}^\mu(u, v)\mathcal{L}_{13}(u)\mathcal{L}_{23}(v) = \mathcal{L}_{23}(v)\mathcal{L}_{13}(u)\mathcal{R}_{12}^\mu(u, v), \tag{3.13}$$

assuming the R-operator $\mathcal{R}_{12}^\mu(u, v)$ and the L-operator $\mathcal{L}_{jk}(u)$ in the form,

$$\mathcal{R}_{12}^\mu(u, v) = \alpha \mathcal{R}_{12}(u - v) + \beta \mathcal{R}_{12}(u + v)(2n_1 - 1), \tag{3.14}$$
$$\mathcal{L}_{jk}(u) = \mathcal{R}_{jk}(u) \exp \left\{ h(u)(2n_j - 1) \right\}. \tag{3.15}$$

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Here α, β and $h(u)$ are functions of the spectral parameters to be specified later.

Comparing eq. (3.13) with eq. (3.12), we get a relation

$$\mathcal{I}_1(u)\mathcal{I}_2(v)\mathcal{R}_{12}^{\mu}(u,v)\mathcal{I}_1(u)^{-1}\mathcal{I}_2(v)^{-1} = \alpha\mathcal{R}_{12}(u-v) + \beta(2n_1-1)\mathcal{R}_{12}(u+v), \qquad (3.16)$$

where

$$\mathcal{I}_j(u) = \exp\{h(u)(2n_j-1)\}. \tag{3.17}$$

For $k \neq 0$ (that is, the genuine XY fermion model), the relation (3.16) gives two conditions,

$$\frac{\beta}{\alpha} = \tanh(h(u) - h(v)) \tag{3.18}$$

$$\frac{\beta d(u+v)}{\alpha d(u-v)} = \tanh(h(v) + h(u)). \tag{3.19}$$

Equations (3.18) and (3.19) give the ratio of α to β and constraints on $h(u)$ and $h(v)$. Substituting (3.8) into (3.19), we find that the constraints are given by

$$\frac{\sinh 2h(u)}{\mathrm{sc}\, 2u} = \frac{\sinh 2h(v)}{\mathrm{sc}\, 2v} = \text{constant}, \tag{3.20}$$

where $\mathrm{sc}\, u = \mathrm{sn}\, u / \mathrm{cn}\, u$. In conclusion, we have proved that the fermionic R-operator

$$\mathcal{R}_{12}^{\mu}(u,v) = \mathcal{R}_{12}(u-v) + \tanh(h(u) - h(v))\mathcal{R}_{12}(u+v)(2n_1-1), \tag{3.21}$$

satisfies the Yang-Baxter relation (3.13) with the L-operators (3.15) under the constraints (3.20) .

From eq. (3.13) we get the global Yang-Baxter relation for the monodromy operator

$$\mathcal{T}_a(u) = \mathcal{L}_{aN}(u)\ldots\mathcal{L}_{a1}(u), \tag{3.22}$$

as

$$\mathcal{R}_{12}^{\mu}(u,v)\mathcal{T}_1(u)\mathcal{T}_2(v) = \mathcal{T}_2(v)\mathcal{T}_1(u)\mathcal{R}_{12}^{\mu}(u,v), \tag{3.23}$$

which leads to a one-parameter family of the commuting transfer operators,

$$[\tau(u), \tau(v)] = 0,$$
$$\tau(u) = \mathrm{Str}_a\, \mathcal{T}_a(u). \tag{3.24}$$

If we identify the constant in eq. (3.20) with $\mu/2$, we recover the Hamiltonian (3.11) by the logarithmic derivative of the transfer operator

$$\mathcal{H} = \tau(0)^{-1}\frac{\mathrm{d}}{\mathrm{d}u}\tau(u)\Big|_{u=0}$$
$$= \mathcal{H}^{\mathrm{XY}} + \mu\sum_{j=1}^{N}\left(n_j - \frac{1}{2}\right). \tag{3.25}$$

Here we have used the relations $h(0) = 0, \quad h'(0) = \mu/2$.

4 Fermionic R-operator for the 1D Hubbard model

The Hamiltonian of the 1D Hubbard model (1.1) consists of two XX fermion models (3.10) for up and down spins with an interaction term between them. The fermionic R-operator without an interaction term is given by

$$\bar{\mathcal{R}}_{jk}(u) = \mathcal{R}_{jk}^{(\uparrow)}(u)\mathcal{R}_{jk}^{(\downarrow)}(u), \tag{4.1}$$

where $\mathcal{R}_{jk}^{(\sigma)}$ ($\sigma =\uparrow\downarrow$) denote the fermionic R-operators for the XX fermion model, i.e.,

$$a(u) = \cos u, \quad b(u) = \sin u, \quad c(u) = 1, \quad d(u) = 0. \tag{4.2}$$

Since both $\mathcal{R}_{jk}^{(\uparrow)}(u)$ and $\mathcal{R}_{jk}^{(\downarrow)}(u)$ satisfy the YBE and the DYBE, the product $\bar{\mathcal{R}}_{jk}(u)$ also satisfies the YBE

$$\bar{\mathcal{R}}_{12}(u - v)\bar{\mathcal{R}}_{13}(u)\bar{\mathcal{R}}_{23}(v) = \bar{\mathcal{R}}_{23}(v)\bar{\mathcal{R}}_{13}(u)\bar{\mathcal{R}}_{12}(u - v) \tag{4.3}$$

and the DYBE

$$\begin{aligned}
\bar{\mathcal{R}}_{12}(u + v)(2n_{1\uparrow} - 1)&(2n_{1\downarrow} - 1)\bar{\mathcal{R}}_{13}(u)\bar{\mathcal{R}}_{23}(v) \\
&= \bar{\mathcal{R}}_{23}(v)\bar{\mathcal{R}}_{13}(u)(2n_{1\uparrow} - 1)(2n_{1\downarrow} - 1)\bar{\mathcal{R}}_{12}(u + v).
\end{aligned} \tag{4.4}$$

A linear combination of (4.3) and (4.4) yields

$$\begin{aligned}
\Big\{\alpha\bar{\mathcal{R}}_{12}(u - v) &+ \beta\bar{\mathcal{R}}_{12}(u + v)(2n_{1\uparrow} - 1)(2n_{1\downarrow} - 1)\Big\}\bar{\mathcal{R}}_{13}(u)\bar{\mathcal{R}}_{23}(v) \\
&= \bar{\mathcal{R}}_{23}(v)\bar{\mathcal{R}}_{13}(u)\Big\{\alpha\bar{\mathcal{R}}_{12}(u - v) + \beta(2n_{1\uparrow} - 1)(2n_{1\downarrow} - 1)\bar{\mathcal{R}}_{12}(u + v)\Big\}.
\end{aligned} \tag{4.5}$$

For a moment, α and β are arbitrary. As in the case of the XY fermion model in an external field, we look for a solution of the Yang-Baxter relation,

$$\mathcal{R}_{12}^{\mathrm{h}}(u, v)\mathcal{L}_{13}(u)\mathcal{L}_{23}(v) = \mathcal{L}_{23}(v)\mathcal{L}_{13}(u)\mathcal{R}_{12}^{\mathrm{h}}(u, v) \tag{4.6}$$

in the form

$$\mathcal{R}_{12}^{\mathrm{h}}(u, v) = \alpha\bar{\mathcal{R}}_{12}(u - v) + \beta\bar{\mathcal{R}}_{12}(u + v)(2n_{1\uparrow} - 1)(2n_{1\downarrow} - 1), \tag{4.7}$$

$$\mathcal{L}_{jk}(u) = \bar{\mathcal{R}}_{jk}(u) \exp\left\{-h(u)(2n_{j\uparrow} - 1)(2n_{j\downarrow} - 1)\right\}. \tag{4.8}$$

Comparing eq. (4.5) with the Yang-Baxter relation (4.6), we get a relation

$$\begin{aligned}
\mathcal{I}_1(u)\mathcal{I}_2(v)&\mathcal{R}_{12}^{\mathrm{h}}(u, v)\mathcal{I}_1(u)^{-1}\mathcal{I}_2(v)^{-1} \\
&= \alpha\bar{\mathcal{R}}_{12}(u - v) + \beta(2n_{1\uparrow} - 1)(2n_{1\downarrow} - 1)\bar{\mathcal{R}}_{12}(u + v),
\end{aligned} \tag{4.9}$$

where

$$\mathcal{I}_j(u) = \exp\{-h(u)(2n_{j\uparrow} - 1)(2n_{j\downarrow} - 1)\}. \tag{4.10}$$

From (4.9), we obtain two conditions

$$\frac{\beta a(u+v)}{\alpha a(u-v)} = -\tanh(h(u) - h(v)), \quad \frac{\beta b(u+v)}{\alpha b(u-v)} = -\tanh(h(u) + h(v)). \tag{4.11}$$

which give the ratio of α to β and constraints on $h(u)$ and $h(v)$. The constraints are given more explicitly by

$$\frac{\sinh 2h(u)}{\sin 2u} = \frac{\sinh 2h(v)}{\sin 2v} = \frac{U}{4}. \tag{4.12}$$

To sum up, we have obtained the fermionic R-operator for the 1D Hubbard model

$$\mathcal{R}_{12}^{\mathrm{h}}(u,v) = \mathcal{R}_{12}^{(\uparrow)}(u-v)\mathcal{R}_{12}^{(\downarrow)}(u-v) - \frac{\cos(u-v)}{\cos(u+v)}\tanh(h(u) - h(v))$$
$$\times \mathcal{R}_{12}^{(\uparrow)}(u+v)\mathcal{R}_{12}^{(\downarrow)}(u+v)(2n_{1\uparrow} - 1)(2n_{1\downarrow} - 1), \tag{4.13}$$

which satisfies the Yang-Baxter relation (4.6). One of the remarkable properties of the fermionic R-operator (4.13) is that its dependence on the the spectral parameters is not a difference type $u - v$. We call it the "non-additive" property. The non-additive property of the fermionic R-operator (4.13) allows us to generalize the Hamiltonian of the 1D Hubbard model (see §5).

From (4.6) we get the global Yang-Baxter relation,

$$\mathcal{R}_{12}^{\mathrm{h}}(u,v)\mathcal{T}_1(u)\mathcal{T}_2(v) = \mathcal{T}_2(v)\mathcal{T}_1(u)\mathcal{R}_{12}^{\mathrm{h}}(u,v) \tag{4.14}$$

for the monodromy operator

$$\mathcal{T}_a(u) = \mathcal{L}_{aN}(u)\ldots\mathcal{L}_{a1}(u). \tag{4.15}$$

Now we define the supertrace $\mathrm{Str}^{\mathrm{h}}$ for the 1D Hubbard model as

$$\mathrm{Str}_a^{\mathrm{h}} X = {}_a\langle 0|X|0\rangle_a - {}_a\langle\uparrow|X|\uparrow\rangle_a - {}_a\langle\downarrow|X|\downarrow\rangle_a + {}_a\langle\downarrow\uparrow|X|\uparrow\downarrow\rangle_a, \tag{4.16}$$

where the auxiliary fermion Fock space is given by

$$c_{a\sigma}|0\rangle_a = 0, \quad |\uparrow\rangle_a = c_{a\uparrow}^\dagger|0\rangle_a, \quad |\downarrow\rangle_a = c_{a\downarrow}^\dagger|0\rangle_a, \quad |\uparrow\downarrow\rangle_a = c_{a\uparrow}^\dagger c_{a\downarrow}^\dagger|0\rangle_a,$$
$$_a\langle 0| = (|0\rangle_a)^\dagger, \quad {}_a\langle\uparrow| = {}_a\langle 0|c_{a\uparrow}, \quad {}_a\langle\downarrow| = {}_a\langle 0|c_{a\downarrow}, \quad {}_a\langle\downarrow\uparrow| = {}_a\langle 0|c_{a\downarrow}c_{a\uparrow}. \tag{4.17}$$

Then the transfer operator of the 1D Hubbard model is defined by

$$\tau(u) = \mathrm{Str}_a^{\mathrm{h}} \mathcal{T}_a(u). \tag{4.18}$$

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From the global Yang-Baxter relation (4.14), we can show the existence of a commuting family of the transfer operators

$$[\tau(u), \tau(v)] = 0, \tag{4.19}$$

which proves the integrability of the model.

Using the relations $h(0) = 0$ and $h'(0) = U/4$, we can obtain the Hamiltonian of the 1D Hubbard model as a logarithmic derivative of the transfer operator,

$$
\begin{aligned}
\mathcal{H} &= -\tau(0)^{-1} \frac{\mathrm{d}}{\mathrm{d}u} \tau(u) \Big|_{u=0} \\
&= -\sum_{j=1}^{N} \sum_{\sigma=\uparrow\downarrow} \mathcal{H}_{j,j+1}^{\mathrm{XX}(\sigma)} + \frac{U}{4} \sum_{j=1}^{N} (2n_{j\uparrow} - 1)(2n_{j\downarrow} - 1),
\end{aligned}
\tag{4.20}
$$

where the PBC for the fermion operators (1.3) is satisfied.

In this way we have proved the integrability of the 1D Hubbard model using the fermionic R-operator. It should be emphasized that we have not used the Jordan-Wigner transformation at all, which changes the boundary condition and the symmetry of the model. [20] We shall see in §6 that the fermionic R-operator (4.13) naturally enjoys the $SO(4)$ symmetry.

5   Yang-Baxter Equation for the New Fermionic R-Operator

In the previous section we have shown that the fermionic R-operator (4.13) satisfies the Yang-Baxter relation (4.6) with the L-operator (4.8). It is natural to expect that the fermionic R-operator itself fulfills the Yang-Baxter equation (YBE),

$$\mathcal{R}_{12}^{\mathrm{h}}(u_1, u_2) \mathcal{R}_{13}^{\mathrm{h}}(u_1, u_3) \mathcal{R}_{23}^{\mathrm{h}}(u_2, u_3) = \mathcal{R}_{23}^{\mathrm{h}}(u_2, u_3) \mathcal{R}_{13}^{\mathrm{h}}(u_1, u_3) \mathcal{R}_{12}^{\mathrm{h}}(u_1, u_2). \tag{5.1}$$

In the case of the coupled spin model, the YBE of the R-matrix was proved in ref. [10] using the tetrahedral Zamolodchikov algebra (TZA). [12, 13] In this section, we introduce the TZA for the fermionic R-operator and proves the YBE (5.1).

The TZA is defined by the following set of relations among operators $\mathcal{L}_{jk}^{0}$ and $\mathcal{L}_{jk}^{1}$,

$$\mathcal{L}_{12}^{a} \mathcal{L}_{13}^{b} \mathcal{L}_{23}^{c} = \sum_{d,e,f=0,1} S_{def}^{abc} \mathcal{L}_{23}^{a} \mathcal{L}_{13}^{b} \mathcal{L}_{12}^{c}, \qquad a, \dots, f = 0, 1, \tag{5.2}$$

where S_{def}^{abc} are some scalar coefficients.

Let us take $\mathcal{L}_{jk}^{(0)}$ and $\mathcal{L}_{jk}^{(1)}$ as

$$
\begin{aligned}
\mathcal{L}_{jk}^{0(\sigma)} &= \mathcal{R}_{jk}^{(\sigma)}(u_j - u_k), \\
\mathcal{L}_{jk}^{1(\sigma)} &= \mathcal{R}_{jk}^{(\sigma)}(u_j + u_k)(2n_{j\sigma} - 1), \qquad \sigma = \uparrow\downarrow,
\end{aligned}
\tag{5.3}
$$

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where $\mathcal{R}_{jk}^{(\uparrow)}(u)$ and $\mathcal{R}_{jk}^{(\downarrow)}(u)$ are the fermionic R-operators for the XX fermion model as before. Then we have found the following relations which give the TZA (5.2),

$$\mathcal{L}_{12}^{0(\sigma)}\mathcal{L}_{13}^{0(\sigma)}\mathcal{L}_{23}^{0(\sigma)} = \mathcal{L}_{23}^{0(\sigma)}\mathcal{L}_{13}^{0(\sigma)}\mathcal{L}_{12}^{0(\sigma)}, \quad \mathcal{L}_{12}^{0(\sigma)}\mathcal{L}_{13}^{1(\sigma)}\mathcal{L}_{23}^{1(\sigma)} = \mathcal{L}_{23}^{1(\sigma)}\mathcal{L}_{13}^{1(\sigma)}\mathcal{L}_{12}^{0(\sigma)} \tag{5.4}$$

$$\mathcal{L}_{12}^{1(\sigma)}\mathcal{L}_{13}^{1(\sigma)}\mathcal{L}_{23}^{0(\sigma)} = \mathcal{L}_{23}^{0(\sigma)}\mathcal{L}_{13}^{1(\sigma)}\mathcal{L}_{12}^{1(\sigma)}, \quad \mathcal{L}_{12}^{1(\sigma)}\mathcal{L}_{13}^{0(\sigma)}\mathcal{L}_{23}^{1(\sigma)} = \mathcal{L}_{23}^{1(\sigma)}\mathcal{L}_{13}^{0(\sigma)}\mathcal{L}_{12}^{1(\sigma)}, \tag{5.5}$$

$$\mathcal{L}_{12}^{1(\sigma)}\mathcal{L}_{13}^{1(\sigma)}\mathcal{L}_{23}^{1(\sigma)} = S_{001}^{111}\mathcal{L}_{23}^{1(\sigma)}\mathcal{L}_{13}^{0(\sigma)}\mathcal{L}_{12}^{0(\sigma)} + S_{010}^{111}\mathcal{L}_{23}^{0(\sigma)}\mathcal{L}_{13}^{1(\sigma)}\mathcal{L}_{12}^{0(\sigma)} + S_{100}^{111}\mathcal{L}_{23}^{0(\sigma)}\mathcal{L}_{13}^{0(\sigma)}\mathcal{L}_{12}^{1(\sigma)}, \tag{5.6}$$

$$\mathcal{L}_{12}^{0(\sigma)}\mathcal{L}_{13}^{0(\sigma)}\mathcal{L}_{23}^{1(\sigma)} = S_{111}^{001}\mathcal{L}_{23}^{1(\sigma)}\mathcal{L}_{13}^{1(\sigma)}\mathcal{L}_{12}^{1(\sigma)} + S_{100}^{001}\mathcal{L}_{23}^{0(\sigma)}\mathcal{L}_{13}^{0(\sigma)}\mathcal{L}_{12}^{1(\sigma)} + S_{010}^{001}\mathcal{L}_{23}^{0(\sigma)}\mathcal{L}_{13}^{1(\sigma)}\mathcal{L}_{12}^{0(\sigma)}, \tag{5.7}$$

$$\mathcal{L}_{12}^{0(\sigma)}\mathcal{L}_{13}^{1(\sigma)}\mathcal{L}_{23}^{0(\sigma)} = S_{111}^{010}\mathcal{L}_{23}^{1(\sigma)}\mathcal{L}_{13}^{1(\sigma)}\mathcal{L}_{12}^{1(\sigma)} + S_{100}^{010}\mathcal{L}_{23}^{0(\sigma)}\mathcal{L}_{13}^{0(\sigma)}\mathcal{L}_{12}^{1(\sigma)} + S_{001}^{010}\mathcal{L}_{23}^{1(\sigma)}\mathcal{L}_{13}^{0(\sigma)}\mathcal{L}_{12}^{0(\sigma)}, \tag{5.8}$$

$$\mathcal{L}_{12}^{1(\sigma)}\mathcal{L}_{13}^{0(\sigma)}\mathcal{L}_{23}^{0(\sigma)} = S_{111}^{100}\mathcal{L}_{23}^{1(\sigma)}\mathcal{L}_{13}^{1(\sigma)}\mathcal{L}_{12}^{1(\sigma)} + S_{010}^{100}\mathcal{L}_{23}^{0(\sigma)}\mathcal{L}_{13}^{1(\sigma)}\mathcal{L}_{12}^{0(\sigma)} + S_{001}^{100}\mathcal{L}_{23}^{1(\sigma)}\mathcal{L}_{13}^{0(\sigma)}\mathcal{L}_{12}^{0(\sigma)}, \tag{5.9}$$

where $\sigma = \uparrow$ or $\downarrow$, and the coefficients S_{def}^{abc} are given by

$$S_{001}^{111} = \frac{\sin(u_1+u_2)\cos(u_1+u_3)}{\cos(u_1-u_2)\sin(u_1-u_3)}, \quad S_{010}^{111} = -\frac{\sin(u_1+u_2)\sin(u_2+u_3)}{\cos(u_1-u_2)\cos(u_2-u_3)},$$

$$S_{100}^{111} = -\frac{\sin(u_2+u_3)\cos(u_1+u_3)}{\cos(u_2-u_3)\sin(u_1-u_3)}, \quad S_{111}^{001} = \frac{\sin(u_1-u_2)\cos(u_1-u_3)}{\cos(u_1+u_2)\sin(u_1+u_3)},$$

$$S_{100}^{001} = \frac{\sin(u_1-u_2)\sin(u_2+u_3)}{\cos(u_1+u_2)\cos(u_2-u_3)}, \quad S_{010}^{001} = \frac{\sin(u_2+u_3)\cos(u_1-u_3)}{\cos(u_2-u_3)\sin(u_1+u_3)},$$

$$S_{111}^{010} = \frac{\sin(u_1-u_2)\sin(u_2-u_3)}{\cos(u_1+u_2)\cos(u_2+u_3)}, \quad S_{100}^{010} = \frac{\sin(u_1-u_2)\cos(u_1+u_3)}{\cos(u_1+u_2)\sin(u_1u_3)},$$

$$S_{001}^{010} = \frac{\sin(u_2-u_3)\cos(u_1+u_3)}{\cos(u_2+u_3)\sin(u_1-u_3)}, \quad S_{111}^{100} = -\frac{\sin(u_2-u_3)\cos(u_1-u_3)}{\cos(u_2+u_3)\sin(u_1+u_3)},$$

$$S_{010}^{100} = \frac{\sin(u_1+u_2)\cos(u_1-u_3)}{\cos(u_1-u_2)\sin(u_1+u_3)}, \quad S_{001}^{100} = -\frac{\sin(u_1+u_2)\sin(u_2-u_3)}{\cos(u_1-u_2)\cos(u_2+u_3)}. \tag{5.10}$$

Note that eq. (5.4) and eq. (5.5) are equivalent to the YBE (4.3) and the DYBE (4.4) respectively. In this sense, the TZA (5.2) can be regarded as a generalization of the YBE and the DYBE.

We also remark that the products $\mathcal{L}_{12}^{a(\sigma)}\mathcal{L}_{13}^{b(\sigma)}\mathcal{L}_{23}^{c(\sigma)}$ $(a, b, c = 0, 1)$ are not linearly independent and satisfy the following relations,

$$\mathcal{L}_{12}^{0(\sigma)}\mathcal{L}_{13}^{0(\sigma)}\mathcal{L}_{23}^{0(\sigma)} = x\mathcal{L}_{12}^{0(\sigma)}\mathcal{L}_{13}^{1(\sigma)}\mathcal{L}_{23}^{1(\sigma)} + y\mathcal{L}_{12}^{1(\sigma)}\mathcal{L}_{13}^{0(\sigma)}\mathcal{L}_{23}^{1(\sigma)} + z\mathcal{L}_{12}^{1(\sigma)}\mathcal{L}_{13}^{1(\sigma)}\mathcal{L}_{23}^{0(\sigma)}, \tag{5.11}$$

$$\mathcal{L}_{12}^{1(\sigma)}\mathcal{L}_{13}^{1(\sigma)}\mathcal{L}_{23}^{1(\sigma)} = x'\mathcal{L}_{12}^{1(\sigma)}\mathcal{L}_{13}^{0(\sigma)}\mathcal{L}_{23}^{0(\sigma)} + y'\mathcal{L}_{12}^{0(\sigma)}\mathcal{L}_{13}^{1(\sigma)}\mathcal{L}_{23}^{0(\sigma)} + z'\mathcal{L}_{12}^{0(\sigma)}\mathcal{L}_{13}^{0(\sigma)}\mathcal{L}_{23}^{1(\sigma)}, \tag{5.12}$$

where

$$x = -\frac{\sin(u_1 - u_3)\cos(u_2 - u_3)}{\sin(u_1 + u_3)\cos(u_2 + u_3)}, \qquad y = \frac{\cos(u_1 - u_2)\cos(u_2 - u_3)}{\cos(u_1 + u_2)\cos(u_2 + u_3)},$$

$$z = \frac{\cos(u_1 - u_2)\cos(u_1 - u_3)}{\cos(u_1 + u_2)\cos(u_1 + u_3)}, \qquad x' = -\frac{\sin(u_1 + u_3)\cos(u_2 + u_3)}{\sin(u_1 - u_3)\cos(u_2 - u_3)},$$

$$y' = \frac{\cos(u_1 + u_2)\cos(u_2 + u_3)}{\cos(u_1 - u_2)\cos(u_2 - u_3)}, \qquad z' = \frac{\sin(u_1 + u_3)\cos(u_1 + u_2)}{\sin(u_1 - u_3)\cos(u_1 - u_2)}. \tag{5.13}$$

These relations (5.4) - (5.13) have been verified using mathematica.

Now taking into account the form of the fermionic R-operator (4.13), we look for a solution of the YBE (5.1) in the form

$$\mathcal{R}_{jk}^{h}(u_j, u_k) = \mathcal{L}_{jk}^{0(\uparrow)}(u_j, u_k)\mathcal{L}_{jk}^{0(\downarrow)}(u_j, u_k) + \alpha_{jk}\mathcal{L}_{jk}^{1(\uparrow)}(u_j, u_k)\mathcal{L}_{jk}^{1(\downarrow)}(u_j, u_k). \tag{5.14}$$

We substitute eq. (5.14) into the YBE (5.1). Using the TZA (5.2) and the relations (5.11) and (5.12), we obtain the conditions

$$\alpha_{13}\sin 2(u_1 + u_3) = \alpha_{12}\sin 2(u_1 + u_2) + \alpha_{23}\sin 2(u_2 + u_3)$$

$$= \frac{1}{\alpha_{12}}\sin 2(u_1 - u_2) - \frac{1}{\alpha_{13}}\sin 2(u_1 - u_3). \tag{5.15}$$

If we assume α_{ij} in the form

$$\alpha_{jk} = -\frac{\cos(u_j - u_k)}{\cos(u_j + u_k)}\tanh(h(u_j) - h(u_k)). \tag{5.16}$$

and impose the constraints

$$\frac{\sinh 2h(u_j)}{\sin 2u_j} = \frac{U}{4}, \qquad\qquad j = 1, 2, 3, \tag{5.17}$$

then one can easily find that the conditions (5.15) are satisfied. In conclusion, the fermionic R-operator for the 1D Hubbard model,

$$\mathcal{R}_{jk}^{h}(u_j, u_k) = \mathcal{R}_{jk}^{(\uparrow)}(u_j - u_k)\mathcal{R}_{jk}^{(\downarrow)}(u_j - u_k) - \frac{\cos(u_j - u_k)}{\cos(u_j + u_k)}\tanh(h(u_j) - h(u_k))$$

$$\times \mathcal{R}_{jk}^{(\uparrow)}(u_j + u_k)\mathcal{R}_{jk}^{(\downarrow)}(u_j + u_k)(2n_{j\uparrow} - 1)(2n_{j\downarrow} - 1), \tag{5.18}$$

is a solution of the YBE (5.1) under the constraints (5.17).

Besides the YBE (5.1), the fermionic R-operator (5.18) has the following properties

$$\mathcal{R}_{jk}^{h}(u, 0) = \frac{1}{\cosh h(u)}\mathcal{L}_{jk}(u), \tag{5.19}$$

$$\mathcal{R}_{jk}^{h}(u_0, u_0) = \mathcal{P}_{jk}, \tag{5.20}$$

$$\mathcal{R}_{jk}^{h}(u_j, u_k)\mathcal{R}_{kj}^{h}(u_k, u_j) = \rho(u_j, u_k)\mathbf{1}, \tag{5.21}$$

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where

$$\rho(u_j, u_k) = \cos^2(u_j - u_k) \left\{ \cos^2(u_j - u_k) - \tanh^2(h(u_j) - h(u_k)) \cos^2(u_j + u_k) \right\}. \tag{5.22}$$

Here the permutation operator is defined by

$$\mathcal{P}_{jk} = \mathcal{P}_{jk}^{(\uparrow)} \mathcal{P}_{jk}^{(\downarrow)}$$
$$\mathcal{P}_{jk}^{(\sigma)} = 1 - (c_{j\sigma}^\dagger - c_{k\sigma}^\dagger)(c_{j\sigma} - c_{k\sigma}), \qquad \sigma = \uparrow\downarrow. \tag{5.23}$$

Due to eq. (5.19), we can recover the Yang-Baxter relation (4.6) by putting $u_3 = 0$ in the YBE (5.1).

Using the R-operator (5.18), we can introduce an inhomogeneous model as

$$\mathcal{T}_a(u, \{u_j\}) = \mathcal{R}_{aN}^{\mathrm{h}}(u, u_N) \dots \mathcal{R}_{aj}^{\mathrm{h}}(u, u_j) \dots \mathcal{R}_{a1}^{\mathrm{h}}(u, u_1), \tag{5.24}$$

where u_j $(j = 1, \dots, N)$ are the inhomogeneous parameters obeying the constraints

$$\frac{\sinh 2h(u_j)}{\sin 2u_j} = \frac{U}{4}, \qquad j = 1, \dots, N. \tag{5.25}$$

Then from the YBE (5.1), we have the global Yang-Baxter relation

$$\mathcal{R}_{12}^{\mathrm{h}}(u, v)\mathcal{T}_1(u, \{u_j\})\mathcal{T}_2(v, \{u_j\}) = \mathcal{T}_2(v, \{u_j\})\mathcal{T}_1(u, \{u_j\})\mathcal{R}_{12}^{\mathrm{h}}(u, v), \tag{5.26}$$

which leads to the commutativity

$$[\tau(u, \{u_j\}), \tau(v, \{u_j\})] = 0. \tag{5.27}$$

Because of the non-additive property of the R-operator (5.18), the simplest choice

$$\mathcal{T}_a(u, u_0) = \mathcal{R}_{aN}^{\mathrm{h}}(u, u_0) \dots \mathcal{R}_{aj}^{\mathrm{h}}(u, u_0) \dots \mathcal{R}_{a1}^{\mathrm{h}}(u, u_0) \tag{5.28}$$

still generalizes eq. (4.15). Actually we can derive a fermionic integrable Hamiltonian which generalize the 1D Hubbard model. Note that $\tau(u_0, u_0)$ is the shift operator regardless of the parameter u_0,

$$\tau(u_0, u_0) = \hat{U}$$
$$\equiv \mathcal{P}_{12}\mathcal{P}_{23} \dots \mathcal{P}_{N-1,N}. \tag{5.29}$$

Then the logarithmic derivative of the generalized transfer operator yields a new fermionic

Hamiltonian as follows,

$$
\mathcal{H} = - \tau(u_0, u_0)^{-1} \frac{\mathrm{d}}{\mathrm{d}u} \tau(u, u_0) \Big|_{u=u_0}
$$

$$
= - \sum_{j=1}^{N} \sum_{\sigma=\uparrow\downarrow} \left(c_{j\sigma}^{\dagger} c_{j+1\sigma} + c_{j+1\sigma}^{\dagger} c_{j\sigma} \right)
$$

$$
+ \frac{U}{4\cosh 2h(u_0)} \sum_{j=1}^{N} \Big\{ \cos^2 u_0 (2n_{j\uparrow} - 1) - \sin^2 u_0 (2n_{j+1\uparrow} - 1)
$$

$$
+ \sin 2u_0 (c_{j+1\uparrow}^{\dagger} c_{j\uparrow} - c_{j\uparrow}^{\dagger} c_{j+1\uparrow}) \Big\}
$$

$$
\times \Big\{ \cos^2 u_0 (2n_{j\downarrow} - 1) - \sin^2 u_0 (2n_{j+1\downarrow} - 1)
$$

$$
+ \sin 2u_0 (c_{j+1\downarrow}^{\dagger} c_{j\downarrow} - c_{j\downarrow}^{\dagger} c_{j+1\downarrow}) \Big\}. \qquad (5.30)
$$

If we take $u_0 = 0$ or $\pi/2$, the Hamiltonian (5.30) reduces to the 1D Hubbard model (1.1). The local higher conserved operators can also be obtained by the formula

$$
I_n = \frac{\mathrm{d}^n}{\mathrm{d}u^n} \ln \left(\tau(u_0, u_0)^{-1} \tau(u, u_0) \right) \Big|_{u=u_0}, \qquad n = 1, 2, \ldots, \qquad (5.31)
$$

where $I_1 = -\mathcal{H}$. The meaning of the insertion of the term $\tau(u_0, u_0)^{-1}$ in (5.31) will be clarified when we discuss the $SO(4)$ symmetry in §6.

Remark that in terms of $\check{\mathcal{R}}_{jk}(u, u_0) \equiv \mathcal{P}_{jk} \mathcal{R}_{jk}(u, u_0)$ we can express the Hamiltonian (5.30) as

$$
\mathcal{H} = - \sum_{j=1}^{N} \frac{\mathrm{d}}{\mathrm{d}u} \check{\mathcal{R}}_{jk}(u, u_0) \Big|_{u=u_0}. \qquad (5.32)
$$

The higher conserved operators I_n are also expressed in terms of $\check{\mathcal{R}}_{jk}(u, u_0)$, though it is very difficult to get their explicit forms.

6 $SO(4)$ Symmetry

The $SO(4)(= [SU(2) \times SU(2)]/Z_2)$ symmetry is one of the most important properties of the Hubbard Hamiltonian (1.1) [14–18]. The two $SU(2)$ come from the spin $SU(2)$

$$
S^+ = \sum_{j=1}^{N} c_{j\uparrow}^{\dagger} c_{j\downarrow}, \quad S^- = \sum_{j=1}^{N} c_{j\downarrow}^{\dagger} c_{j\uparrow}, \quad S^z = \frac{1}{2} \sum_{j=1}^{N} (n_{j\uparrow} - n_{j\downarrow}) \qquad (6.1)
$$

and the charge $SU(2)$ (η-pairing $SU(2)$)

$$
\eta^+ = \sum_{j=1}^{N} (-1)^j c_{j\uparrow}^{\dagger} c_{j\downarrow}^{\dagger}, \quad \eta^- = \sum_{j=1}^{N} (-1)^j c_{j\downarrow} c_{j\uparrow}, \quad \eta^z = \frac{1}{2} \sum_{j=1}^{N} (n_{j\uparrow} + n_{j\downarrow} - 1). \qquad (6.2)
$$

We assume that the lattice has an even number of sites under the PBC (1.3). Then the six generators above commute with the Hamiltonian (1.1).

In ref. [19,20], the $SO(4)$ symmetry of the fermionic transfer matrix was discussed based on the fermionic formulation by Olmedilla *et al..* [7] In the following we discuss the $SO(4)$ symmetry of the fermionic R-operator and the generalized transfer operator (5.28) using the similar discussion (c.f. ref. [32]). For this purpose, we introduce the local generators of the $SO(4)$ algebra as follows,

$$S_j^+ = c_{j\uparrow}^\dagger c_{j\downarrow}, \quad S_j^- = c_{j\downarrow}^\dagger c_{j\uparrow}, \quad S_j^z = \frac{1}{2}(n_{j\uparrow} - n_{j\downarrow}),$$

$$\eta_j^+ = c_{j\uparrow}^\dagger c_{j\downarrow}^\dagger, \quad \eta_j^- = c_{j\downarrow} c_{j\uparrow}, \quad \eta_j^z = \frac{1}{2}(n_{j\uparrow} + n_{j\downarrow} - 1). \tag{6.3}$$

The $SU(2)$ symmetry of the fermionic R-operator for the 1D Hubbard model (5.18) is represented by the following commutation relations,

$$\left[\mathcal{R}_{jk}^{\mathrm{h}}(u, v), S_j^+ + S_k^+ \right] = 0, \quad \left[\mathcal{R}_{jk}^{\mathrm{h}}(u, v), S_j^- + S_k^- \right] = 0,$$

$$\left[\mathcal{R}_{jk}^{\mathrm{h}}(u, v), S_j^z + S_k^z \right] = 0. \tag{6.4}$$

It is easy to see that the relations (6.4) extend to the $SU(2)$ symmetry of the monodromy operator

$$\mathcal{T}_a(u, u_0) = \mathcal{R}_{aN}^{\mathrm{h}}(u, u_0) \dots \mathcal{R}_{a1}^{\mathrm{h}}(u, u_0)$$

as

$$\left[\mathcal{T}_a(u, u_0), S_a^\alpha + S^\alpha \right] = 0, \qquad \alpha = \pm, z. \tag{6.5}$$

Taking the supertrace Str_a of (6.5), we obtain

$$\left[\tau(u, u_0), S^\alpha \right] = 0, \qquad \alpha = \pm, z, \tag{6.6}$$

which shows that the transfer operator $\tau(u, u_0)$ is the spin $SU(2)$ invariant.

Next we consider the charge $SU(2)$ symmetry. For the generator η_j^z, the same relation as (6.4) holds,

$$\left[\mathcal{R}_{jk}^{\mathrm{h}}(u, v), \eta_j^z + \eta_k^z \right] = 0, \tag{6.7}$$

which extends to the symmetry of the monodromy operator and the transfer operator as,

$$\left[\mathcal{T}_a(u, u_0), \eta_a^z + \eta^z \right] = 0, \quad \left[\tau(u, u_0), \eta^z \right] = 0. \tag{6.8}$$

As for the other generators $\eta_j^\pm$, we do not have the relations like (6.4). Instead, we have discovered the following anti-commuting relation for $\eta_j^\pm$,

$$\left\{ \mathcal{R}_{jk}^{\mathrm{h}}(u, v), \eta_j^\pm - \eta_k^\pm \right\} = 0. \tag{6.9}$$

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The local relation (6.9) extends to the identities for the monodromy operator,

$$\left[\mathcal{T}_a(u,u_0),\eta_a^{\pm}\right] + \left\{\mathcal{T}_a(u,u_0),\eta^{\pm}\right\} = 0, \tag{6.10}$$

which can be proved as follows,

$$
\begin{aligned}
\mathcal{T}_a&(u,u_0)\left(\eta_a^{\pm} + \eta^{\pm}\right) \\
&= \mathcal{R}_{aN}^{\mathrm{h}}(u,u_0)\dots\mathcal{R}_{a2}^{\mathrm{h}}(u,u_0)\mathcal{R}_{a1}^{\mathrm{h}}(u,u_0)(\eta_a^{\pm} - \eta_1^{\pm} + \eta_2^{\pm}\dots + \eta_N^{\pm}) \\
&= \mathcal{R}_{aN}^{\mathrm{h}}(u,u_0)\dots\mathcal{R}_{a2}^{\mathrm{h}}(u,u_0)(-\eta_a^{\pm} + \eta_1^{\pm} + \eta_2^{\pm}\dots + \eta_N^{\pm})\mathcal{R}_{a1}^{\mathrm{h}}(u,u_0) \\
&= \dots \\
&= (\eta_a^{\pm} + \eta_1^{\pm} - \eta_2^{\pm}\dots - \eta_N^{\pm})\mathcal{R}_{aN}^{\mathrm{h}}(u,u_0)\dots\mathcal{R}_{a2}^{\mathrm{h}}(u,u_0)\mathcal{R}_{a1}^{\mathrm{h}}(u,u_0) \\
&= (\eta_a^{\pm} - \eta^{\pm})\mathcal{T}_a(u,u_0).
\end{aligned} \tag{6.11}
$$

Here we have used the fact that N is even.

Taking the supertrace of (6.10), we obtain

$$\left\{\tau(u,u_0),\eta^{\pm}\right\} = 0. \tag{6.12}$$

Namely the transfer operator anti-commutes with the generators $\eta^{\pm}$. Thus contrary to the naive anticipation, the transfer operator itself does not commute with the generators of the charge $SU(2)$ symmetry. However a combination $\tau(u_0,u_0)^{-1}\tau(u,u_0)$ commutes with $\eta^{\pm}$

$$\left[\tau(u_0,u_0)^{-1}\tau(u,u_0),\eta^{\pm}\right] = 0, \tag{6.13}$$

due to the relation

$$\left\{\tau(u_0,u_0),\eta^{\pm}\right\} = 0. \tag{6.14}$$

Of course, $\tau(u_0,u_0)^{-1}\tau(u,u_0)$ commutes with the other generators of the $SO(4)$ algebra. Since we have defined the local conserved operators by eq. (5.31),

$$I_n = \frac{\mathrm{d}^n}{\mathrm{d}u^n}\ln\left(\tau(u_0,u_0)^{-1}\tau(u,u_0)\right)\Big|_{u=u_0}, \qquad n = 1,2,\dots \tag{6.15}$$

it follows that they are all $SO(4)$ invariant. In particular, we have proved that a generalized Hubbard model (5.30) has the $SO(4)$ symmetry. It is also possible to prove the $SO(4)$ invariance of the Hamiltonian (5.30) directly using the expression (5.32) and the commutation relations

$$
\begin{aligned}
&\left[\check{\mathcal{R}}_{jk}^{\mathrm{h}}(u,v), S_j^{\alpha} + S_k^{\alpha}\right] = 0, &&\alpha = \pm, z, \\
&\left[\check{\mathcal{R}}_{jk}^{\mathrm{h}}(u,v), \eta_j^{z} + \eta_k^{z}\right] = 0, &&\left[\check{\mathcal{R}}_{jk}^{\mathrm{h}}(u,v), \eta_j^{\pm} - \eta_k^{\pm}\right] = 0.
\end{aligned} \tag{6.16}
$$

where $\check{\mathcal{R}}_{jk}^{\mathrm{h}}(u,v) = \mathcal{P}_{jk}\mathcal{R}_{jk}^{\mathrm{h}}(u,v)$ as before.

The relation (6.14) means that the shift operator $\hat{U} = \tau(u_0, u_0)$ anticummutes with $\eta^{\pm}$

$$\hat{U}\eta^{\pm} = -\eta^{\pm}\hat{U}. \tag{6.17}$$

If we define the lattice momentum operator $\hat{\Pi}$ [19,32] by

$$\hat{U} = \exp i\hat{\Pi}, \tag{6.18}$$

the relation (6.17) implies that

$$\hat{\Pi}\eta^{\pm} - \eta^{\pm}\hat{\Pi} = \pi\eta^{\pm}. \tag{6.19}$$

This indicates that the generators $\eta^{\pm}$ changes the momentum eigenvalue by π. [15,19] Note that the lattice momentum operator $\hat{\Pi}$ is defined in (6.18) mod 2π.

The lattice momentum operator $\hat{\Pi}$ is also a conserved operator

$$\left[\hat{\Pi}, I_n\right] = 0, \tag{6.20}$$

but it is not contained in $\{I_n\}$. Actually $\hat{\Pi}$ is a non-local operator (we refer ref. [19] for the detailed discussions on $\hat{\Pi}$). Furthermore the relation (6.19) suggests that it does not have the $SO(4)$ symmetry.

7   Concluding Remarks

In this paper, we have presented a new fermionic approach for the integrability of the 1D Hubbard model. The central object in our approach is the fermionic R-operator, which we introduced recently to discuss the integrability of the XXZ fermion model. We have shown that the fermionic R- operator can be generalized to the XYZ fermion model. It gives an operator solution of the Yang-Baxter equation (YBE). We also considered the decorated Yang-Baxter equation (DYBE) for the fermionic R-operator and found that the free-fermion condition is necessary. In other words, the fermionic R-operator related to the XY fermion model fulfills the YBE and the DYBE.

The 1D Hubbard model can be regarded as two XX fermion models with an interaction term. Using the YBE and the DYBE for the XX fermion model, we have established a new kind of the Yang-Baxter relation for the 1D Hubbard model. This gives an alternative proof for the integrability of the 1D Hubbard model. We have obtained the fermionic R-operator for the 1D Hubbard model and confirmed that it satisfies the YBE. The connection between the fermionic R-operator and the graded Yang-Baxter relation for the 1D Hubbard model [7] will be discussed in a separate paper. [33]

From the fermionic R-operator of the 1D Hubbard model, we have constructed a new fermionic transfer operator. Because of the non-additive property of the R-operator, the transfer operator generates a fermionic integrable Hamiltonian (5.30) which generalize the 1D Hubbard model.

One of the advantages of our fermionic approach is that we can discuss the $SO(4)$ symmetry at the level of the (local) fermionic R-operator, which immediately leads to the $SO(4)$ symmetry of all the local conserved operators derived from the transfer operator. It is known that the $SO(4)$ symmetry of the 1D Hubbard model is extended to the Yangian symmetry on the infinite lattice [34,35]. It is interesting to investigate the relation between the fermionic R-operator and the Yangian symmetry.

Acknowledgment

APPENDIX A: A Simple Derivation of the Fermionic R-Operator

The R-matrix of the XYZ spin models in terms of the Pauli spin matrix (σ_j^+, σ_j^- and σ_j^z) is given by

$$
\begin{aligned}
R_{jk}(u) =& a(u)\frac{1+\sigma_j^z\sigma_k^z}{2} + b(u)\frac{1-\sigma_j^z\sigma_k^z}{2} + c(u)(\sigma_j^+\sigma_k^- + \sigma_k^+\sigma_j^-) \\
& + d(u)(\sigma_j^+\sigma_k^+ + \sigma_j^-\sigma_k^-),
\end{aligned}
\tag{A.1}
$$

where the Boltzmann weights $a(u), b(u), c(u)$ and $d(u)$ are the same as (2.5). The R-matrix (A.1) satisfies the Yang-Baxter equation

$$
R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v).
\tag{A.2}
$$

We like to obtain the fermionic R-operator (2.2) from the R-matrix (A.1) by means of the Jordan-Wigner transformation [30]

$$
\begin{aligned}
\sigma_j^- &= \exp\left\{ i\pi \sum_{k=1}^{j-1} n_k \right\} c_j, \\
\sigma_j^+ &= \exp\left\{ -i\pi \sum_{k=1}^{j-1} n_k \right\} c_j^\dagger,
\end{aligned}
\tag{A.3}
$$

However it is not possible to apply the Jordan-Wigner transformation directly to (A.1) because of the non-local term $R_{13}(u)$ in the Yang-Baxter equation (A.2). For this reason, we consider an equivalent R-matrix $\check{R}_{jk}(u)$

$$
\begin{aligned}
\check{R}_{jk}(u) \equiv& P_{jk}R_{jk}(u) \\
=& a(u)\frac{1+\sigma_j^z\sigma_k^z}{2} + c(u)\frac{1-\sigma_j^z\sigma_k^z}{2} + b(u)(\sigma_j^+\sigma_k^- + \sigma_j^-\sigma_k^+) \\
& + d(u)(\sigma_j^+\sigma_k^+ + \sigma_j^-\sigma_k^-),
\end{aligned}
\tag{A.4}
$$

which satisfies the Yang-Baxter equation in the form

$$\check{R}_{12}(u-v)\check{R}_{23}(u)\check{R}_{12}(v) = \check{R}_{23}(v)\check{R}_{12}(u)\check{R}_{23}(u-v). \tag{A.5}$$

Here P_{jk} is the permutation matrix

$$P_{jk} = \frac{1 + \sigma_j^z \sigma_k^z}{2} + \sigma_j^+ \sigma_k^- + \sigma_j^- \sigma_k^+. \tag{A.6}$$

Now we can apply the Jordan-Wigner transformation (A.3) to (A.4) and (A.5). Then we obtain a fermionic solution

$$\begin{aligned}\check{\mathcal{R}}_{jk}(u) =&\, a(u)\{n_j n_k + (1-n_j)(1-n_k)\} + c(u)\{n_j(1-n_k) + (1-n_j)n_k\} \\ &+ b(u)(c_j^\dagger c_k + c_k^\dagger c_j) + d(u)(c_j^\dagger c_k^\dagger - c_j c_k),\end{aligned} \tag{A.7}$$

satisfying

$$\check{\mathcal{R}}_{12}(u-v)\check{\mathcal{R}}_{23}(u)\check{\mathcal{R}}_{12}(v) = \check{\mathcal{R}}_{23}(v)\check{\mathcal{R}}_{12}(u)\check{\mathcal{R}}_{23}(u-v). \tag{A.8}$$

In the derivation, the following transformation laws are of particular importance,

$$\begin{aligned}\sigma_j^+ \sigma_{j+1}^+ &\longrightarrow c_j^\dagger c_{j+1}^\dagger, \\ \sigma_j^- \sigma_{j+1}^- &\longrightarrow -c_j c_{j+1}.\end{aligned} \tag{A.9}$$

Strictly speaking, $\check{\mathcal{R}}_{jk}(u)$ is defined in (A.7) only for $k = j+1$. However we extends its definition to arbitrary j and k by the formula (A.7).

Finally multiplying the products of the fermionic permutation operators (2.7)

$$\mathcal{P}_{12}\mathcal{P}_{13}\mathcal{P}_{23} = \mathcal{P}_{23}\mathcal{P}_{13}\mathcal{P}_{12}, \tag{A.10}$$

to (A.7) from the left, we obtain

$$\mathcal{R}_{12}(u-v)\mathcal{R}_{13}(u)\mathcal{R}_{23}(v) = \mathcal{R}_{23}(v)\mathcal{R}_{13}(u)\mathcal{R}_{12}(u-v), \tag{A.11}$$

with

$$\begin{aligned}\mathcal{R}_{jk}(u) =&\, \mathcal{P}_{jk}\check{\mathcal{R}}_{jk}(u) \\ =&\, a(u)\{-n_j n_k + (1-n_j)(1-n_k)\} + b(u)\{n_j(1-n_k) + (1-n_j)n_k\} \\ &+ c(u)(c_j^\dagger c_k + c_k^\dagger c_j) - d(u)(c_j^\dagger c_k^\dagger + c_j c_k).\end{aligned} \tag{A.12}$$

This is identical to the fermionic R-operator (2.2). In this way we can obtain the fermionic R-operator (2.2) from the R-matrix of the XYZ spin model.

Just in the same way, the fermionic R-operator for the 1D Hubbard model can be obtained from the R-matrix of the coupled spin model (1.5).

References

[1] E. H. Lieb and F. Y. Wu: Phys. Rev. Lett. **20** (1968) 1445.

[2] V. E. Korepin and F. H. L. Eßler: *Exactly Solvable Models of Strongly Correlated Electrons* (World Scientific, Singapore, 1994).

[3] B. S. Shastry: Phys. Rev. Lett. **56** (1986) 1529.

[4] B. S. Shastry: Phys. Rev. Lett. **56** (1986) 2453.

[5] B. S. Shastry: J. Stat. Phys. **50** (1988) 57.

[6] M. Wadati, E. Olmedilla and Y. Akutsu: J. Phys. Soc. Jpn. **56** (1987) 1340.

[7] E. Olmedilla and M. Wadati and Y. Akutsu: J. Phys. Soc. Jpn. **56** (1987) 2298.

[8] E. Olmedilla and M. Wadati: Phys. Rev. Lett. **60** (1988) 1595.

[9] P. Jordan and E. Wigner: Z. Phys. **47** (1928) 631.

[10] M. Shiroishi and M. Wadati: J. Phys. Soc. Jpn. **64** (1995) 57.

[11] M. Shiroishi and M. Wadati: J. Phys. Soc. Jpn. **64** (1995) 2795.

[12] I. G. Korepanov: Comm. Math. Phys. **154** (1993) 85.

[13] M. Shiroishi and M. Wadati: J. Phys. Soc. Jpn. **64** (1995) 4598.

[14] O. J. Heilmann and E. H. Lieb: Ann. N.Y. Acad. Sci. **172** (1971) 584.

[15] C. N. Yang: Phys. Rev. Lett. **63** (1989) 2144.

[16] C. N. Yang and S. C. Zhang: Mod. Phys. Lett. **B4** (1990) 759.

[17] M. Pernici: Europhys. Lett. **12** (1990) 75.

[18] I. Affleck: in *Physics, Geometry and Topology*, ed. H. C. Lee (Plenum Press, New York, 1990) p.1.

[19] F. Göhmann and S. Murakami: J. Phys. A: Math. Gen. **30** (1997) 5269.

[20] M. Shiroishi, H. Ujino and M. Wadati: *"SO(4) Symmetry of the Transfer Matrix for the One-Dimensional Hubbard Model"* (cond-mat/**9708064**), to appear in J. Phys. A: Math. Gen. (1998).

[21] P. B. Ramos and M. J. Martins: J. Phys. A: Math. Gen. **30** (1997) L195.

[22] M.J. Martins and P. B. Ramos: *"Quantum Inverse Scattering Method for the Hubbard-like Models"* (solv-int/**9712014**) (1997).

[23] C. Destri and T. Segalini: Nucl. Phys. B **455** (1995) 759.

[24] Y. Umeno, M. Shiroishi and M. Wadati: *"Fermionic R-operator for the Fermion Chain Model"*, to appear in J. Phys. Soc. Jpn. (1998).

[25] R. J. Baxter: *Exactly Solved Models in Statistical Mechanics*, (Academic Press, London, 1982) p 210.

[26] P. P. Kulish and E. K. Sklyanin: in *Integrable Quantum Field Theories*, ed. J. Hietarinta and C. Montonen (Springer Verlag, Berlin, 1981) p. 61.

[27] E. K. Sklyanin: in *Quantum Group and Quantum Integrable Systems*, ed. M. L. Ge (World Scientific, Singapore, 1992) p. 63.

[28] V. E. Korepin, N. M. Bogoliubov and A. G. Izergin: *Quantum Inverse Scattering Method and Correlation Functions*, (Cambridge University Press, Cambridge, 1993).

[29] C. Gomez, M. R. Altaba and G. Sierra: *Quantum Groups in Two-dimensional Physics* (Cambridge University Press, Cambridge, 1996) p. 58.

[30] E. Lieb, T. Schultz, and D. Mattis: Ann. Phys. **16** (1961) 407.

[31] P. Pfeuty: Ann. Phys. **57** (1970) 79.

[32] L. A. Takhtajan and L. D. Faddeev: J. Sov. Math. **24** 241.

[33] M. Shiroishi and Y. Umeno: in preparation.

[34] D. B. Uglov and V. E. Korepin: Phys. Lett. **190 A** (1994) 238.

[35] S. Murakami and F. Göhmann: Phys. Lett. **227 A** (1997) 216.